Mock exam Q & A.

11. 26.

Zhipeng Xue.

Q1 $\sum_{n=1}^{\infty} \frac{e^{\lambda n}}{n^{1+\lambda}}$ converges if and only if $\lambda \in I$. $I = ?$

Cauchy's criterion

$$a_n = \frac{e^{\lambda n}}{n^{1+\lambda}}$$

$$\sqrt[n]{a_n} = \frac{e^{\lambda}}{n^{\frac{1+\lambda}{n}}} \longrightarrow e^{\lambda} \quad \text{when } n \to \infty.$$

(here we use $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$).

You can also try D'Alambert.

try $\lambda = 0$ and -1

Q2. $u_0 = \sqrt{3}$, $\lim u_n = \lim \sqrt{3u_{n-1}}$, $\lim_{n\to\infty} u_n = ?$

if the limit exists, $\lim_{n\to\infty} u_n = u$,

$$u = \sqrt{3u}$$

$$\Rightarrow u^2 - 3u = 0$$

$$u = 0 \text{ or } 3$$

$u_n > 0$ by induction. So $u \neq 0$.

Approach 1.

prove that $u_n < 3$ by induction.

$$0 < u_n < 3$$

besides, $u_{n+1}^2 - u_n^2$

$$= 3u_n - u_n^2$$

$$= u_n(3 - u_n). > 0.$$

$$u_{n+1} > u_n.$$

$$(u_n) \nearrow.$$

$$\lim_{n \to \infty} u_n = 3$$

Approach 2 Compute u_n

$$\log u_{n+1} = \frac{1}{2} \log(3u_n)$$

$$= \frac{1}{2}\left(\log(\cancel{3}u_n) + \log(3)\right).$$

Construct a geometric sequence

$$\log u_{n+1} + c = \frac{1}{2}(\log(u_n) + c).$$

$$= \frac{1}{2}\log(u_n) - \frac{1}{2}c$$

$$-\frac{1}{2}c = \frac{1}{2}\log(3).$$

$$c = -\log(3).$$

$$\log u_{n+1} - \log 3 = \frac{1}{2}(\log(u_n) - \log 3)$$

$$\log u_n - \log 3 = \frac{1}{2^n} (\log u_0 - \log 3).$$

$$\log u_n = \underbrace{\log 3 - \frac{1}{2^{n+1}} \log 3}_{} \xrightarrow{n \to +\infty} \log 3$$

$$u_n \to 3$$

Q 3 ✓

Q 4. $$\frac{|z|}{z} = \frac{z^2}{4\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)}$$

$$z = ?$$

Polar representation

$$z = r \cdot e^{i\theta}, \quad r \geq 0.$$

$$0 \leq \theta < 2\pi.$$

$$|z| = \frac{z^3}{4\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)}.$$

$$r = \frac{r^3 e^{i3\theta}}{4 e^{i\frac{\pi}{3}}}.$$

$$\Rightarrow r^2 e^{i3\theta} = 4 e^{i\frac{\pi}{3}}.$$

$$\Rightarrow \begin{cases} r^2 = 4 \\ 3\theta = \frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \begin{cases} r = 2 \\ \theta = \frac{\pi}{9} + \frac{2}{3}k\pi \end{cases}$$

$$\theta = \frac{\pi}{9} \text{ or } \frac{7\pi}{9} \text{ or } \frac{13\pi}{9}.$$

Q5.

$$a_n = \frac{1}{2} a_{n-1} + \frac{1}{2} \quad (1).$$

$$a_n - c = \frac{1}{2}(a_{n-1} - c). \quad (2).$$

$$a_n = \frac{1}{2} a_{n-1} + \frac{1}{2}c$$

$$c = 1$$

$$a_n - 1 = \frac{1}{2^n}(a_0 - 1)$$

$$a_n = 1 + \frac{1}{2^n}(a_0 - 1).$$

Q6. ✓

Q7. $x_n = e^{2\sqrt{n^2+1} - n}$

$$\lim_{n\to\infty} x_n = ?$$

$$\lim_{n\to\infty} x_n = e^{\lim_{n\to\infty} \overset{\infty}{2\sqrt{n^2+1}} - \overset{\infty}{n}}$$

$$= e^{\lim_{n\to\infty} \frac{(2\sqrt{n^2+1}-n)(2\sqrt{n^2+1}+n)}{2\sqrt{n^2+1}+n}} \qquad \boxed{\infty - \infty \quad \frac{\infty}{\infty}}$$

$$= e^{\lim_{n\to\infty} \frac{3n^2+4}{2\sqrt{n^2+1}+n}}$$

$$= e^{\lim_{n\to\infty} \frac{3+\frac{4}{n^2}}{2\sqrt{\frac{1}{n^2}+\frac{1}{n^4}}+\frac{1}{n^3}}} \qquad \frac{3}{0}$$

$$= \infty$$

Q8. ✓

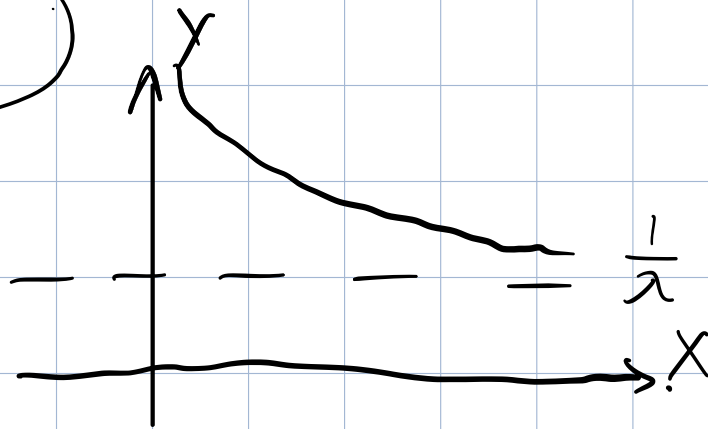
Q13. $a_n = \left(\frac{\lambda+n}{\lambda n}\right)^n$

for every $\lambda \in \mathbb{R}^*$ s.t. (a_n) c.v.

do we have $\lim_{n\to\infty} a_n = 0$?

the base $\frac{\lambda + n}{\lambda n}$

$$y = \frac{\lambda + x}{\lambda x}$$



1° $0 < \lambda < 1$, $\frac{1}{\lambda} > 1$

$$\left(\frac{\lambda + n}{\lambda n}\right)^n > \left(\frac{1}{\lambda}\right)^n.$$

2° $\lambda = 1$

$$a_n = \left(\frac{1+n}{n}\right)^n = \boxed{\left(1+\frac{1}{n}\right)^n \to e.}$$

when $n \to +\infty$.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

$$= e^{\lim\limits_{n\to\infty} n\log\left(1+\frac{1}{n}\right)}$$

$$= e^{\lim\limits_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right) - \log(1+0)}{\frac{1}{n} - 0}}$$

$$\left(\log(1+x)\right)' = \frac{1}{1+x}$$

$$= e^{\lim\limits_{n\to\infty} \frac{1}{1+\xi}} \qquad \xi \in \left(0, \frac{1}{n}\right)$$

$$= e$$

3° $\lambda > 1$

$$\left(\frac{\lambda+n}{\lambda n}\right)^n \approx q^n \quad 0<q<1, \text{ when } n \text{ is large.}$$

$$\to 0$$

$$\boxed{\begin{aligned} &\lim_{x\to 0} \frac{\sin x}{x} = 1. \\ &\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \end{aligned}}$$

two important limits